

# The $\pi\pi$ $S$ -wave scattering lengths

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## Abstract

We match the known chiral perturbation theory representation of the  $\pi\pi$  scattering amplitude to two loops with a phenomenological description that relies on the Roy equations. On this basis, the corrections to Weinberg's low energy theorems for the  $S$ -wave scattering lengths are worked out to second order in the expansion in powers of the quark masses. The resulting predictions,  $a_0^0 = 0.220 \pm 0.005$ ,  $a_0^2 = -0.0444 \pm 0.0010$ , contain remarkably small uncertainties and thus allow a very sensitive experimental test of the hypothesis that the quark condensate is the leading order parameter of the spontaneously broken chiral symmetry.

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## 1. Introduction

In the chiral limit, where the masses of the two lightest quarks are turned off, QCD acquires an exact  $SU(2) \times SU(2)$  symmetry. We rely on the standard picture, where it is assumed that this symmetry is broken spontaneously and that the quark condensate  $\langle 0 | \bar{u}u | 0 \rangle$  represents the leading order parameter. The quark masses act as symmetry breaking parameters, which equip the Goldstone bosons – the pions – with a mass<sup>1</sup>:  $M_\pi^2 = 2mB + O(m^2)$ , where  $B$  stands for the value of  $|\langle 0 | \bar{u}u | 0 \rangle|/F_\pi^2$  in the chiral limit.

Since Goldstone bosons can interact only if they carry momentum, the  $\pi\pi$   $S$ -wave scattering lengths vanish in the chiral limit. Hence these quantities represent a sensitive probe of the symmetry breaking generated by the quark masses. Weinberg's low energy theorems [1] state that their values are related to the pion mass, which also represents a symmetry breaking effect:

$$a_0^0 = \frac{7 M_\pi^2}{32 \pi F_\pi^2} + O(m^2) \quad , \quad a_0^2 = -\frac{M_\pi^2}{16 \pi F_\pi^2} + O(m^2) \quad . \quad (1)$$

The relations hold to leading order in the expansion in powers of  $m$ . The corrections of order  $m^2$  were worked out in [2]. As an example, we quote the result for the combination  $2a_0^0 - 5a_0^2$ , which is particularly simple:

$$2a_0^0 - 5a_0^2 = \frac{3M_\pi^2}{4\pi F_\pi^2} \left\{ 1 + \frac{1}{3} M_\pi^2 \langle r^2 \rangle_s + \frac{41}{12} \xi \right\} + O(m^3) \quad . \quad (2)$$

$\langle r^2 \rangle_s$  is the mean square radius of the scalar form factor and  $\xi$  measures the pion mass in units of the scale set by  $4\pi F_\pi \simeq 1 \text{ GeV}$ :

$$\xi = \left( \frac{M_\pi}{4\pi F_\pi} \right)^2 = 0.01445 \quad . \quad (3)$$

With the evaluation of the chiral perturbation series to two loops described in ref. [3], the low energy expansion of the scattering amplitude is now known to next-to-next-to-leading order. The purpose of the present paper is to analyze the consequences for the scattering lengths.

## 2. Chiral representation of the scattering amplitude

The two-loop representation yields the first three terms in the low energy expansion of the partial waves:

$$t_\ell^I(s) = t_\ell^I(s)_2 + t_\ell^I(s)_4 + t_\ell^I(s)_6 + O(p^8) \quad . \quad (4)$$

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<sup>1</sup>We disregard isospin breaking and set  $m_u = m_d = m$ . In the numerical work, we identify  $M_\pi$  with the mass of the charged pion and use  $F_\pi = 92.4 \text{ MeV}$ .

Since inelastic reactions start showing up only at  $O(p^8)$ , unitarity implies

$$\text{Im } t_\ell^I(s) = \sigma(s) |t_\ell^I(s)|^2 + O(p^8) \quad , \quad \sigma(s) = (1 - 4M_\pi^2/s)^{\frac{1}{2}} \quad . \quad (5)$$

The condition fixes the imaginary parts of the two-loop amplitude in terms of the one-loop representation. At leading order, the scattering amplitude is linear in the Mandelstam variables, so that only the S- and P-waves are different from zero. Unitarity therefore implies that, up to and including  $O(p^6)$ , only these partial waves develop an imaginary part. Accordingly, the chiral representation of the scattering amplitude can be written in the form [4]

$$A(s, t, u) = C(s, t, u) + 32\pi \left\{ \frac{1}{3} U^0(s) + \frac{3}{2} (s - u) U^1(t) + \frac{3}{2} (s - t) U^1(u) + \frac{1}{2} U^2(t) + \frac{1}{2} U^2(u) - \frac{1}{3} U^2(s) \right\} + O(p^8) \quad , \quad (6)$$

where  $C(s, t, u)$  is a crossing symmetric polynomial,

$$C(s, t, u) = c_1 + s c_2 + s^2 c_3 + (t - u)^2 c_4 + s^3 c_5 + s (t - u)^2 c_6 \quad . \quad (7)$$

The functions  $U^0(s)$ ,  $U^1(s)$  and  $U^2(s)$  describe the “unitarity corrections” associated with  $s$ -channel isospin  $I = 0, 1, 2$ , respectively. In view of  $\text{Im } t_\ell^I(s)_6 \propto s^3$ , several subtractions are needed for the dispersive representation of these functions to converge. We subtract at  $s = 0$  and set

$$U^I(s) = \frac{s^{4-\epsilon_I}}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\sigma(s') t^I(s')_2 \{t^I(s')_2 + 2 \text{Re } t^I(s')_4\}}{s'^{4-\epsilon_I} (s' - 4M_\pi^2)^{\epsilon_I} (s' - s)} \quad . \quad (8)$$

The subtraction constants are collected in  $C(s, t, u)$ . As only the  $S$ - and  $P$ -waves enter, we have dropped the lower index,  $\{t^0, t^1, t^2\} = \{t_0^0, t_1^1, t_0^2\}$ . For kinematic reasons, the integrand of the  $P$ -wave differs from the one of the  $S$ -waves:  $\{\epsilon_0, \epsilon_1, \epsilon_2\} = \{0, 1, 0\}$ . It is straightforward to check that the result of the two-loop calculation [3] is indeed of this structure.

### 3. Effective coupling constants

In the following, the two-loop result for the polynomial part of the amplitude plays a key role. It involves the coupling constants occurring in the derivative expansion of the effective Lagrangian,  $\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \dots$ . The corresponding formulae, which specify how the coefficients  $c_1, \dots, c_6$  depend on the quark masses, can be worked out from the two-loop representation given in ref. [3] (for explicit expressions, see [5]). These formulae, in particular contain Weinberg’s low energy theorems, which in this language state that the leading terms in the expansion of the first two coefficients are fixed by  $M_\pi$  and  $F_\pi$ :  $c_1 = -M_\pi^2/F_\pi^2 + \dots$ ,  $c_2 = 1/F_\pi^2 + \dots$ . At first order, the constants  $\ell_1, \ell_2, \ell_3, \ell_4$  from  $\mathcal{L}_4$  [6] enter, and at second order, the chiral representation of the scattering amplitude involves the couplings  $r_1, \dots, r_6$  from  $\mathcal{L}_6$  [7]. We need to distinguish two categories of coupling constants:

- a. *Terms that survive in the chiral limit.* Four of the coupling constants that enter the two-loop representation of the scattering amplitude belong to this category:  $\ell_1, \ell_2, r_5, r_6$ .
- b. *Symmetry breaking terms.* The corresponding vertices are proportional to a power of the quark mass and involve the coupling constants  $\ell_3, \ell_4, r_1, r_2, r_3, r_4$ .

The constants of the first category show up in the momentum dependence of the scattering amplitude, so that these couplings may be determined phenomenologically. The symmetry breaking terms, on the other hand, specify the dependence of the amplitude on the quark masses. Since these cannot be varied experimentally, information concerning the second category of coupling constants can only be obtained from sources other than  $\pi\pi$  scattering. The constants  $r_n$  from  $\mathcal{L}_6$  only generate tiny effects, so that crude theoretical estimates suffice, but the couplings  $\ell_3$  and  $\ell_4$  from  $\mathcal{L}_4$  do play an important role and we now briefly discuss the information that we will be using for these.

The crucial parameter that distinguishes the standard framework from the one proposed in ref. [8] is  $\ell_3$ . This coupling constant determines the first order correction in the Gell-Mann-Oakes-Renner-relation:  $M_\pi^2 = 2Bm \left\{ 1 - \frac{1}{2}\xi\bar{\ell}_3 + O(\xi^2) \right\}$ . The value of  $\ell_3$  is not known accurately. Numerically, however, a significant change in the prediction for the scattering lengths can only arise if the crude estimate in ref. [6],

$$\bar{\ell}_3 = 2.9 \pm 2.4, \quad (9)$$

should turn out to be entirely wrong. We do not make an attempt at reducing the uncertainty in  $\ell_3$  within the standard framework, but will explicitly indicate the sensitivity to this coupling constant.

Chiral symmetry implies that the coupling constant  $\ell_4$  also shows up in the expansion of the scalar radius in powers of the quark masses [2]:

$$\langle r^2 \rangle_s = \frac{3}{8\pi^2 F_\pi^2} \left\{ \bar{\ell}_4 - \frac{13}{12} + \xi \Delta_r + O(\xi^2) \right\}. \quad (10)$$

As pointed out in ref. [9], the scalar radius can be determined on the basis of a dispersive evaluation of the scalar form factor. We have repeated that calculation with the information about the phase shift  $\delta_0^0(s)$  obtained in ref. [10]. In view of the strong final state interaction in the  $S$ -wave, the scalar radius is significantly larger than the electromagnetic one,  $\langle r^2 \rangle_{e.m.} = 0.439 \pm 0.008 \text{ fm}^2$  [11]. The result,

$$\langle r^2 \rangle_s = 0.61 \pm 0.04 \text{ fm}^2, \quad (11)$$

confirms the value given in ref. [9] and is consistent with earlier estimates of the low energy constant  $\ell_4$  based on the symmetry breaking seen in  $F_K/F_\pi$  or on the decay  $K \rightarrow \pi\ell\nu$  [12], but is considerably more accurate. Since the chiral

representation of the scalar form factor is known to two loops [13], the dependence of the correction  $\Delta_r$  on the quark masses is also known. In addition to  $\ell_1, \dots, \ell_4$ , the explicit expression involves a further term,  $r_{S2}$ , from  $\mathcal{L}_6$  [7]. In the following, we use this representation to eliminate the parameter  $\ell_4$  in favour of the scalar radius.

#### 4. Low energy theorems

We now show that chiral symmetry implies two relations among the coefficients  $c_1, \dots, c_4$ . For this purpose we consider the combinations

$$C_1 \equiv F_\pi^2 \{c_2 + 4M_\pi^2(c_3 - c_4)\} , \quad C_2 \equiv \frac{F_\pi^2}{M_\pi^2} \{-c_1 + 4M_\pi^4(c_3 - c_4)\} . \quad (12)$$

Chiral symmetry implies that, if the quark masses are turned off, both  $C_1$  and  $C_2$  tend to 1. The contributions from  $c_3$  and  $c_4$  ensure that the first order corrections only involve the symmetry breaking couplings  $\ell_3$  and  $\ell_4$ . Eliminating  $\ell_4$  in favour of the scalar radius, the low energy theorems take the form

$$\begin{aligned} C_1 &= 1 + \frac{M_\pi^2}{3} \langle r^2 \rangle_s + \frac{23\xi}{420} + \xi^2 \Delta_1 + O(\xi^3) , \\ C_2 &= 1 + \frac{M_\pi^2}{3} \langle r^2 \rangle_s + \frac{\xi}{2} \left\{ \bar{\ell}_3 - \frac{17}{21} \right\} + \xi^2 \Delta_2 + O(\xi^3) . \end{aligned} \quad (13)$$

At first nonleading order,  $C_1$  is fully determined by the contribution from the scalar radius, while  $C_2$  also contains a contribution from  $\ell_3$ . Inserting the values  $\langle r^2 \rangle_s = 0.61 \text{ fm}^2$  and  $\bar{\ell}_3 = 2.9$  and ignoring the two-loop corrections  $\Delta_1, \Delta_2$ , we obtain  $C_1 = 1.103$ ,  $C_2 = 1.117$ . The value of  $C_2$  differs little from  $C_1$  – as stated above, the estimate (9) implies that the contributions from  $\ell_3$  are very small. The size of the two-loop corrections will be discussed later on, when the phenomenological information about the coefficients  $c_1, \dots, c_4$  has been sorted out.

#### 5. Phenomenological representation of the scattering amplitude

Our predictions for the scattering lengths are based on the comparison of the chiral representation with the one that follows from analyticity and unitarity alone. As shown by Roy [14], the fixed- $t$  dispersion relations can be written in such a form that they express the  $\pi\pi$  scattering amplitude in terms of the imaginary parts in the physical region of the  $s$ -channel. The resulting representation for  $A(s, t, u)$  contains two subtraction constants, which may be identified with the scattering lengths  $a_0^0$  and  $a_0^2$ . Unitarity converts this representation into a set of coupled integral equations, which we recently examined in detail [10]. The upshot of that analysis is that  $a_0^0$  and  $a_0^2$  are the essential low energy parameters: Once these are known, the available experimental data determine the behaviour of the

$\pi\pi$  scattering amplitude at low energies to within remarkably small uncertainties. In the present context, the main result of interest is that the representation allows us to determine the imaginary parts of the partial waves in terms of  $a_0^0$  and  $a_0^2$ . Since the resulting representation is based on the available experimental information, we refer to it as the phenomenological representation.

The branch cut generated by the imaginary parts of the partial waves with  $\ell \geq 2$  starts manifesting itself only at  $O(p^8)$ . Accordingly, we may expand the corresponding contributions to the dispersion integrals into a Taylor series of the momenta. The singularities due to the imaginary parts of the  $S$ - and  $P$ -waves, on the other hand, show up already at  $O(p^4)$  – these cannot be replaced by a polynomial. The corresponding contributions to the amplitude are of the same structure as the unitarity corrections and also involve three functions of a single variable. We subtract the relevant dispersion integrals in the same manner as for the chiral representation:

$$\bar{W}^I(s) = \frac{s^{4-\epsilon_I}}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\text{Im } t^I(s')}{s'^{4-\epsilon_I}(s - 4M_\pi^2)^{\epsilon_I}(s' - s)} . \quad (14)$$

Since all other contributions can be replaced by a polynomial, the phenomenological amplitude takes the form

$$\begin{aligned} A(s, t, u) = & 16\pi a_0^2 + \frac{4\pi}{3M_\pi^2} (2a_0^0 - 5a_0^2) s + \bar{P}(s, t, u) \\ & + 32\pi \left\{ \frac{1}{3} \bar{W}^0(s) + \frac{3}{2}(s - u) \bar{W}^1(t) + \frac{3}{2}(s - t) \bar{W}^1(u) \right. \\ & \left. + \frac{1}{2} \bar{W}^2(t) + \frac{1}{2} \bar{W}^2(u) - \frac{1}{3} \bar{W}^2(s) \right\} + O(p^8) . \end{aligned} \quad (15)$$

We have explicitly displayed the contributions from the subtraction constants  $a_0^0$  and  $a_0^2$ . The term  $\bar{P}(s, t, u)$  is a crossing symmetric polynomial

$$\bar{P}(s, t, u) = \bar{p}_1 + \bar{p}_2 s + \bar{p}_3 s^2 + \bar{p}_4 (t - u)^2 + \bar{p}_5 s^3 + \bar{p}_6 s(t - u)^2 . \quad (16)$$

Its coefficients can be expressed in terms of integrals over the imaginary parts of the partial waves. We do not list the explicit expressions here, but refer to [5]. In the following, the essential point is that the coefficients  $\bar{p}_1, \dots, \bar{p}_6$  can be determined phenomenologically.

## 6. Matching the two representations

We now show that, in their common domain of validity, the two representations of the scattering amplitude specified above agree, provided the parameters occurring therein are properly matched. The key observation is that, in the integrals (14), only the region where  $s'$  is of order  $p^2$  matters for the comparison of the two representations. The remainder generates contributions to the amplitude that are most of order  $p^8$ . Moreover, for small values of  $s'$ , the quantities  $\text{Im } t_\ell^I(s')$  are given

by the one-loop representation, except for contributions that again only manifest themselves at  $O(p^8)$ . This implies that the differences between the functions  $\bar{W}^I(s)$  and  $U^I(s)$  are beyond the accuracy of the two-loop representation [4]. Hence the two descriptions agree if and only if the polynomial parts do,

$$C(s, t, u) = 16\pi a_0^2 + \frac{4\pi}{3M_\pi^2} (2a_0^0 - 5a_0^2) s + \bar{P}(s, t, u) + O(p^8) . \quad (17)$$

Since the main uncertainties in the coefficients of the polynomial  $\bar{P}(s, t, u)$  arise from their sensitivity to the scattering lengths  $a_0^0, a_0^2$ , the above relations essentially determine the coefficients  $c_1, \dots, c_6$  in terms of these two observables. The same then also holds for the quantities  $C_1, C_2$  defined in eq. (12). The corresponding low energy theorems for  $a_0^0$  and  $a_0^2$  are of the form

$$a_0^0 = \frac{7M_\pi^2 C_0}{32\pi F_\pi^2} + M_\pi^4 \alpha_0 + O(m^4), \quad a_0^2 = -\frac{M_\pi^2 C_2}{16\pi F_\pi^2} + M_\pi^4 \alpha_2 + O(m^4), \quad (18)$$

with  $C_0 \equiv \frac{1}{7}(12 C_1 - 5 C_2)$ . The terms  $\alpha_0, \alpha_2$  stand for integrals over the imaginary parts of the partial waves that can be worked out from the available experimental information. Since the behaviour of the imaginary parts near threshold is sensitive to the scattering lengths we are looking for, the same applies to these integrals. In the narrow range of interest, the dependence is very well described by

$$\begin{aligned} M_\pi^4 \alpha_0 &= .04478 + .30\Delta a_0^0 - .37\Delta a_0^2 + .5(\Delta a_0^0)^2 - 1.2\Delta a_0^0 \Delta a_0^2 + 1.8(\Delta a_0^2)^2, \\ M_\pi^4 \alpha_2 &= .0055 + .023\Delta a_0^0 - .095\Delta a_0^2 + .01(\Delta a_0^0)^2 - .12\Delta a_0^0 \Delta a_0^2 + .66(\Delta a_0^2)^2, \end{aligned}$$

with  $\Delta a_0^0 \equiv a_0^0 - 0.225$ ,  $\Delta a_0^2 \equiv a_0^2 + 0.03706$ .

## 7. Results for $a_0^0$ and $a_0^2$

The representation (18) splits the correction to Weinberg's leading order formulae into two parts: a correction factor  $C_n$ , which at first nonleading order only involves the scalar radius and the coupling constant  $\ell_3$ , and a term  $\alpha_n$  that can be determined on phenomenological grounds.

Inserting the one-loop prediction for  $C_1, C_2$  in the relations (18) and solving for  $a_0^0, a_0^2$ , we obtain the following first order results:

$$a_0^0 = 0.2195, \quad a_0^2 = -0.0446, \quad 2a_0^2 - 5a_0^0 = 0.662. \quad (19)$$

The two-loop corrections  $\Delta_1$  and  $\Delta_2$  involve the coupling constants  $\ell_1, \ell_2, \ell_3$ , the scalar radius, as well as the terms  $r_1, \dots, r_4, r_{S2}$  from  $\mathcal{L}_6$ . The size of the contributions from the latter may be estimated with the resonance model described in refs. [3, 13]. The constants  $\ell_1, \ell_2$  can then be determined numerically with the phenomenological values of  $c_3$  and  $c_4$ . The resulting two-loop corrections for the scattering lengths are very small. The numerical result is sensitive to the value

of the scale  $\mu$  at which the renormalized coupling constants  $r_n^r(\mu)$  are assumed to be saturated by the resonance contributions. For  $500 \text{ MeV} < \mu < 1 \text{ GeV}$ , the corrections vary in the range  $-0.001 < \xi^2 \Delta_1 < 0.003$ ,  $-0.004 < \xi^2 \Delta_2 < 0.001$ . In the following, we use the resonance model at the scale  $\mu = M_\rho$  and take the above range as an estimate for the uncertainties to be attached to the two-loop corrections.

To estimate the errors due to the phenomenological input, we vary the imaginary parts within the range discussed in ref. [10], where the Roy equations are used to determine the behaviour of the  $S$ - and  $P$ -waves below 800 MeV. If the  $S$ -wave scattering lengths are held fixed, the variations in the values of  $\alpha_0$ ,  $\alpha_2$  are dominated by those in the input used for the phases at 800 MeV. The corresponding uncertainty in  $\{a_0^0, a_0^2, 2a_0^0 - 5a_0^2\}$  amounts to  $\{\pm 0.9, \pm 0.2, \pm 1\} \cdot 10^{-3}$ .

We conclude that the uncertainties are dominated by those in  $\ell_3$  and  $\langle r^2 \rangle_s$ . Adding the remaining sources of error up, we obtain<sup>2</sup>

$$\begin{aligned} a_0^0 &= 0.220 \pm 0.001 - 0.0017 \Delta \ell_3 + 0.027 \Delta_{r^2}, \\ a_0^2 &= -0.0444 \pm 0.0003 - 0.0004 \Delta \ell_3 - 0.004 \Delta_{r^2}, \end{aligned} \quad (20)$$

with  $\bar{\ell}_3 = 2.9 + \Delta \ell_3$ ,  $\langle r^2 \rangle_s = 0.61 \text{ fm}^2 (1 + \Delta_{r^2})$ . Inserting the estimates (9), (11), we arrive at our final result:

$$\begin{aligned} a_0^0 &= 0.220 \pm 0.005, & a_0^2 &= -0.0444 \pm 0.0010, \\ 2a_0^0 - 5a_0^2 &= 0.663 \pm 0.006, & a_0^0 - a_0^2 &= 0.265 \pm 0.004. \end{aligned} \quad (21)$$

## 8. Discussion

The truncation of the chiral perturbation series represents an inherent limitation of our calculation. We have shown, however, that the corrections of  $O(p^6)$  barely change the predictions for  $a_0^0$  and  $a_0^2$  obtained at  $O(p^4)$ . For this reason, we expect the contributions from yet higher orders to be entirely negligible.

The rapid convergence of the series is a virtue of the specific method used to match the chiral and phenomenological representations. To demonstrate this, we briefly discuss the alternative approach used in ref. [2, 3], where the results for the various scattering lengths and effective ranges are obtained by directly evaluating the chiral representation of the scattering amplitude at threshold. We instead express the amplitude in terms of three functions of a single variable  $s$  and match the coefficients of the Taylor expansion at  $s = 0$  – in this language, the approach of ref. [2, 3] amounts to a matching at threshold. It is straightforward to work out the chiral representation at threshold with the values of the effective coupling constants that we find with our method. Truncating the series at  $O(p^4)$ , we obtain  $\{a_0^0, a_0^2, 2a_0^0 - 5a_0^2\} = \{0.200, -0.0445, 0.624\}$ , in agreement with the

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<sup>2</sup> The tiny errors given in eq.(20) merely indicate the noise seen in our calculation – we do not claim to describe the scattering amplitude to that accuracy (compare section 14.1 of [10]).



values<sup>3</sup> of ref. [2]:  $\{0.20 \pm 0.01, -0.042 \pm 0.002, 0.614 \pm 0.018\}$ . At threshold, the terms of  $O(p^6)$  are by no means negligible: They take the values obtained at  $O(p^4)$  into  $\{0.215, -0.0445, 0.653\}$ , in agreement with the results of ref. [3, 15, 16]. These numbers describe the expansion of the scattering lengths in powers of the quark masses<sup>4</sup> to order  $m^3$ . In the case of  $a_0^0$ , for instance, the numerical values to order  $m$ ,  $m^2$  and  $m^3$  are  $a_0^0 = 0.159, 0.200$  and  $0.215$ , respectively – these correspond to the diamonds shown in the figure.

The reason why the straightforward expansion of the scattering lengths in powers of the quark masses converges rather slowly is that these represent the values of the amplitude at threshold, that is at the place where the branch cut required by unitarity starts. The truncated chiral representation does not describe that singularity well enough, particularly at one loop, where the relevant imaginary parts stem from the tree level approximation. The matching must be done in such a manner that the higher order effects are small. In contrast to a matching at threshold – that is, to the straightforward expansion of the scattering lengths – our method fulfills this criterion remarkably well: We are using the expansion in powers of the quark masses only for the coefficients  $C_1$  and  $C_2$ , while the curvature generated by the unitarity cut is evaluated phenomenologically. Solving eq.(18) with the expansion of these coefficients to order 1,  $m$  and  $m^2$ , we obtain a much more rapidly convergent series:  $a_0^0 = 0.197, 0.2195, 0.220$ .

If the effective coupling constants are the same, the only difference between the two approaches is the one between the functions  $\bar{W}^I(s)$  and  $U^I(s)$ . In particular, the results for  $a_0^0$ ,  $a_0^2$  only differ because the numerical values of  $\bar{W}^I(s)$  and  $U^I(s)$  at  $s = 4M_\pi^2$  are not the same. As mentioned above, the difference between the two sets of functions affects the scattering amplitude only at  $O(p^8)$  and beyond. Numerically, however, it is not irrelevant which one of the two is used to describe the effects generated by the unitarity cuts: While the functions  $\bar{W}^I(s)$  account for the imaginary parts of the  $S$ - and  $P$ -waves to the accuracy to which these are known, the quantities  $U^I(s)$  represent a comparatively crude approximation, obtained by evaluating the imaginary parts with the one-loop representation.

## 9. Conclusion

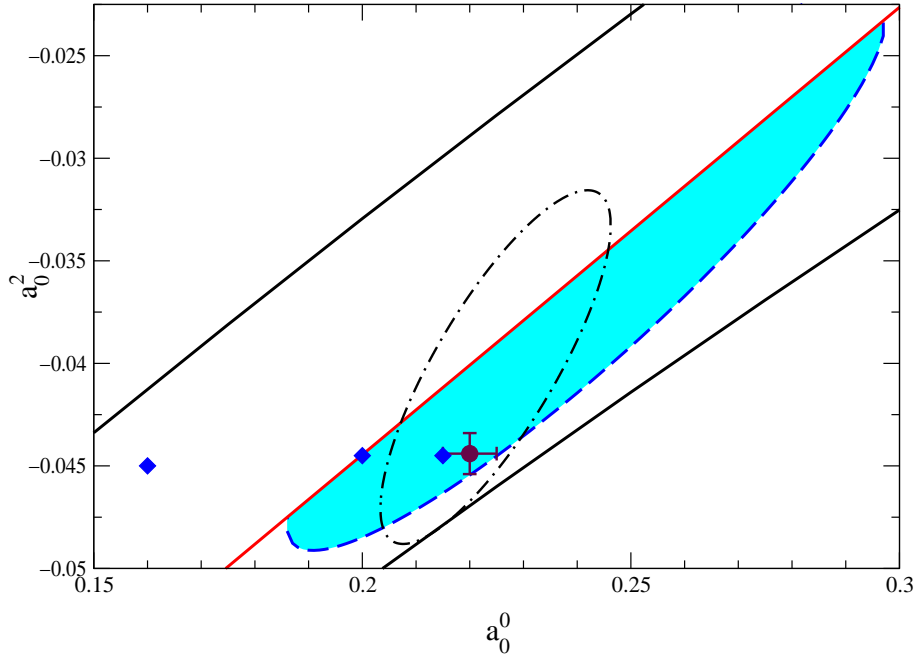
Our analysis relies on two ingredients: the evaluation of the chiral perturbation series for the  $\pi\pi$  scattering amplitude to two loops [3] and the phenomenological representation obtained in ref. [10] by solving the Roy equations. We have shown

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<sup>3</sup>As stated in [2], the error bars given only measure the accuracy to which the first order corrections can be calculated; they do not include an estimate of contributions due to higher order terms. The small numerical differences arise partly from the manner in which the coupling constants  $\bar{\ell}_1, \bar{\ell}_2$  are determined, partly from the values used for  $F_\pi$  – the old number, 93.3 MeV, does not account for radiative corrections.

<sup>4</sup>More precisely, the expansion parameter is the physical pion mass and, in lieu of the coupling constant  $F$ , the physical decay constant is held fixed.

that the comparison of the two descriptions allows us to predict the  $S$ -wave scattering lengths at the 2–3% level of accuracy.



Scattering lengths: theory versus experiment. The shaded region represents the intersection of the domains allowed by the old data and by the Olsson sum rule, see [10] for details. The ellipse indicates the impact of the new, preliminary  $K_{e4}$  data [18]. The cross shows the result of the present paper. The three diamonds illustrate the convergence of the chiral perturbation series at threshold (see text). The one at the left corresponds to Weinberg's leading order formulae. The two black solid lines are the boundaries of the universal band [10].

We emphasize that our result (21) relies on the standard picture, according to which the quark condensate represents the leading order parameter of the spontaneously broken symmetry. The scenario investigated in ref. [8] concerns the possibility that the Gell-Mann-Oakes Renner formula fails, the second term in the expansion  $M_\pi^2 = 2Bm \left\{ 1 - \frac{1}{2} \xi \bar{\ell}_3 + O(\xi^2) \right\}$  being of the same numerical order of magnitude or even larger than the first. Note that for this to happen, the value of  $|\bar{\ell}_3|$  must exceed the estimate (9) by more than an order of magnitude.

The constraints imposed on  $a_0^0$ ,  $a_0^2$  by the available experimental information are shown in the figure. The ellipse represents the 68 % confidence level contour obtained by combining the new, preliminary  $K_{e4}$  data [18] with earlier experimental results. Concerning the value of  $a_0^0$ , the ellipse corresponds to the range  $0.2 < a_0^0 < 0.25$ . The representation (20) shows that this range only puts very weak limits on the value of  $\bar{\ell}_3$ . The precision data on the reaction  $\pi N \rightarrow \pi\pi N$  near threshold [19] provide an independent measurement of  $a_0^0$ . Unfortunately, however, the systematic errors of the

Chew-Low extrapolation that underlies this determination do not appear to be under good control [20] – for a detailed discussion, we refer to [21].

The figure shows that the values of  $a_0^2$  and  $a_0^0$  are strongly correlated. The correlation also manifests itself in the Olsson sum rule [17], which according to ref. [10] leads to  $2a_0^0 - 5a_0^2 = 0.663 \pm 0.021 + 1.13\Delta a_0^0 - 1.01\Delta a_0^2$ , in perfect agreement with our result in eq. (21). Note, however, that this combination is not sensitive to  $\ell_3$  – accurate experimental information in the threshold region is needed to perform a thorough test of the theoretical framework that underlies our calculation. We are confident that the forthcoming results from Brookhaven [18], CERN [22, 23] and Frascati [24] will provide such a test.

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